

ON THE EIGENVALUES OF p -ADIC CURVATURE

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ABSTRACT. We determine the maximal eigenvalue of the p -adic curvature transformations on Bruhat-Tits buildings, and we give an essentially optimal upper bound on the minimal non-zero eigenvalue of these transformations.

1. STATEMENT OF THE RESULTS

Let \mathcal{K} be a non-archimedean locally compact field with finite residue field of order q . Let G be an almost simple linear algebraic group defined over \mathcal{K} of \mathcal{K} -rank $\ell + 1$. Let \mathfrak{T} be the Bruhat-Tits building associated with $G(\mathcal{K})$ [4]. This is an infinite, locally finite, contractible simplicial complex of dimension $\ell + 1$. Let X be the link of a vertex of \mathfrak{T} . X is a finite simplicial complex of dimension ℓ , which is a building in the sense of Tits [3]. In [7], Garland defined a certain combinatorial Laplace operator Δ acting on the i -cochains $C^i(X)$, $0 \leq i \leq \ell - 1$; see Definition 2.3. \mathfrak{T} can be realized as the skeleton of a non-archimedean symmetric space [1, Ch. 5], and from this point of view the operators Δ are the non-archimedean analogues of curvature transformations of riemannian symmetric spaces. Denote by $m^i(X)$ the minimal non-zero eigenvalue of Δ acting on $C^i(X)$. By a rather ingenious argument, Garland proved that for any $\varepsilon > 0$ there is a constant $q(\varepsilon, \ell)$ depending only on ε and ℓ such that if $q > q(\varepsilon, \ell)$ then $m^i(X) \geq \ell - i - \varepsilon$. Denote by $M^i(X)$ the maximal eigenvalue of Δ . The main result of this paper is the following (see Theorems 2.21 and 2.22):

Theorem 1.1. $M^i(X) = \ell + 1$ and $m^i(X) \leq \ell - i$.

In fact we prove this result for an arbitrary finite building X . Note that our estimate on $m^i(X)$ is the best possible estimate which does not depend on q . Based on some explicit calculations, we also propose a conjectural description of the behavior of all the eigenvalues of Δ as $q \rightarrow \infty$; see Conjecture 3.1.

The method of our proof is based on a modification of Garland's original arguments. The results in [7] are stated for buildings. On the other hand, as is nicely explained in [2], part of the argument in [7] works for quite general simplicial complexes. In §2 we follow [2].

The main application of Garland's estimate on $m^i(X)$ is a vanishing result for the cohomology groups of discrete cocompact subgroups of $G(\mathcal{K})$; see §3.1. This vanishing theorem plays an important role in many problems arising in representation theory and arithmetic geometry. Incidentally, our explicit calculations of the eigenvalues of Laplace operators indicate that, despite the hope expressed in [7], Garland's method is not powerful enough to prove the vanishing theorem unconditionally, i.e., without a restriction on q being sufficiently large.

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2. PROOFS

2.1. Simplicial complexes. We start by fixing the terminology and notation related to simplicial complexes.

A *simplicial complex* is a collection X of finite nonempty sets, such that if s is an element of X , so is every nonempty subset of s . The element s of X is called a *simplex* of X ; its *dimension* is $|s| - 1$. Each nonempty subset of s is called a *face* of s . A simplex of dimension i will usually be referred to as i -simplex. The *dimension* $\dim(X)$ of X is the largest dimension of one of its simplices (or is infinite if there is no such largest dimension). A subcollection of X that is itself a complex is called a *subcomplex* of X . The *vertices* of the simplex s are the one-point elements of the set s .

Let s be a simplex of X . The *star* of s in X , denoted $\text{St}(s)$, is the subcomplex of X consisting of the union of all simplices of X having s as a face. The *link* of s , denoted $\text{Lk}(s)$, is the subcomplex of $\text{St}(s)$ consisting of the simplices which are disjoint from s . If one thinks of $\text{St}(s)$ as the “unit ball” around s in X , then $\text{Lk}(s)$ is the “unit sphere” around s .

A specific ordering of the vertices of s up to an even permutation is called an *orientation* of s . An *oriented simplex* is a simplex s together with an orientation of s . Denote the set of i -simplices by $\widehat{S}_i(X)$, and the set of oriented i -simplices by $S_i(X)$. We will denote the vertices $\widehat{S}_0(X) = S_0(X)$ of X also by $\text{Ver}(X)$. For $s \in S_i(X)$, $\bar{s} \in S_i(X)$ denotes the same simplex but with opposite orientation. An \mathbb{R} -valued *i -cochain* on X is a function f from the set of oriented i -simplices of X to \mathbb{R} , such that $f(s) = -f(\bar{s})$. Such functions are also called *alternating*. The i -cochains naturally form a \mathbb{R} -vector space which is denoted $C^i(X)$. If $i < 0$ or $i > \dim(X)$, we let $C^i(X) = 0$.

2.2. Laplace operators. From now on we assume that X is a finite n -dimensional complex such that

(1_X) Each simplex of X is a face of some n -simplex.

For $s \in S_i(X)$, let $w(s)$ be the number of (non-oriented) n -simplices containing s . In view of (1_X), $w(s) \neq 0$ for any s .

Lemma 2.1. *Let $\sigma \in S_i(X)$ be fixed. Then*

$$\sum_{\substack{s \in \widehat{S}_{i+1}(X) \\ \sigma \subset s}} w(s) = (n - i) \cdot w(\sigma).$$

Proof. Given an n -simplex t such that $\sigma \subset t$ there are exactly $(n - i)$ simplices s of dimension $(i + 1)$ such that $\sigma \subset s \subset t$. Hence in the sum of the lemma we count every n -simplex containing σ exactly $(n - i)$ times. \square

Define a positive-definite pairing on $C^i(X)$ by

$$(2.1) \quad (f, g) := \sum_{s \in \widehat{S}_i(X)} w(s) \cdot f(s) \cdot g(s),$$

where $f, g \in C^i(X)$ and in $w(s) \cdot f(s) \cdot g(s)$ we choose some orientation of s . (This is well-defined since both f and g are alternating.)

Define the *coboundary*, a linear transformation $d : C^i(X) \rightarrow C^{i+1}(X)$, by

$$(2.2) \quad (df)([v_0, \dots, v_{i+1}]) = \sum_{j=0}^{i+1} (-1)^j f([v_0, \dots, \hat{v}_j, \dots, v_{i+1}]),$$

where $[v_0, \dots, v_{i+1}] \in S_{i+1}(X)$ and the symbol \hat{v}_j means that the vertex v_j is to be deleted from the array.

Let $s = [v_0, \dots, v_i] \in S_i(X)$ and $v \in \text{Ver}(X)$. If the set $\{v, v_0, \dots, v_i\}$ is an $(i+1)$ -simplex of X , then we denote by $[v, s] \in S_{i+1}(X)$ the oriented simplex $[v, v_0, \dots, v_i]$. Define a linear transformation $\delta : C^i(X) \rightarrow C^{i-1}(X)$ by

$$(2.3) \quad (\delta f)(s) = \sum_{\substack{v \in \text{Ver}(X) \\ [v, s] \in S_i(X)}} \frac{w([v, s])}{w(s)} f([v, s]).$$

In (2.2) and (2.3), by convention, an empty sum is assumed to be 0. One easily checks that δ is the adjoint of d with respect to (2.1):

Lemma 2.2. *If $f \in C^i(X)$ and $g \in C^{i+1}(X)$, then $(df, g) = (f, \delta g)$.*

Definition 2.3. The Laplace operator on $C^i(X)$ is the linear operator $\Delta = \delta d$.

Since Δ is self-adjoint with respect to the pairing (2.1), and for any $f \in C^i(X)$, $(\Delta f, f) = (df, df) \geq 0$, Δ is diagonalizable and its eigenvalues are non-negative real numbers.

Remark 2.4. The Laplace operator in [7, Def. 3.15] is defined as $\delta d + d\delta$. What we denote by Δ in this paper is denoted by Δ^+ in *loc. cit.* When X is the link of a vertex in a Bruhat-Tits building, Garland calls Δ^+ the p -adic curvature; see [7, p. 400].

2.3. Garland's method. For $v \in \text{Ver}(X)$ let ρ_v be the linear transformation on $C^i(X)$ defined by:

$$(\rho_v f)(s) = \begin{cases} f(s) & \text{if } v \in s; \\ 0 & \text{otherwise.} \end{cases}$$

Since any i -simplex has $(i+1)$ -vertices, for $f \in C^i(X)$ we have the obvious equality

$$(2.4) \quad \sum_{v \in \text{Ver}(X)} \rho_v f = (i+1)f.$$

We also have the following obvious lemma:

Lemma 2.5.

- (1) $\rho_v \rho_v = \rho_v$;
- (2) For $f \in C^i(X)$ and $g \in C^i(X)$, $(\rho_v f, g) = (f, \rho_v g)$.

Let d_v and δ_v be the linear operators d and δ acting on the cochains of the finite simplicial complex $\text{Lk}(v)$, and let $\Delta_v := \delta_v d_v$. Note that $\text{Lk}(v)$ is an $(n-1)$ -dimensional complex satisfying condition (1_X) . For $f, g \in C^i(X)$ define their inner product on $\text{Lk}(v)$ by

$$(2.5) \quad (f, g)_v := \sum_{s \in \hat{S}_i(\text{Lk}(v))} w_v(s) \cdot f(s) \cdot g(s),$$

where $w_v(s)$ is the number of $(n-1)$ -simplices in $\text{Lk}(v)$ containing s . This is simply the pairing (2.1) of the restrictions of f and g to $\text{Lk}(v)$.

Lemma 2.6. *If $f \in C^i(X)$, then*

$$i \cdot (\Delta f, f) + (n-i)(f, f) = \sum_{v \in \text{Ver}(X)} (\Delta \rho_v f, \rho_v f).$$

Proof. See [2, Lem. 1.3]. In the proof it is crucial that the inner product (\cdot, \cdot) on $C^i(X)$ is defined using the weights $w(s)$. \square

Corollary 2.7. *Let $f \in C^i(X)$. If there is a positive real number Λ such that*

$$(\Delta \rho_v f, \rho_v f) \leq \Lambda \cdot (\rho_v f, f)$$

for all $v \in \text{Ver}(X)$, then

$$i \cdot (\Delta f, f) \leq (\Lambda \cdot (i+1) - (n-i)) (f, f).$$

Proof. This follows from Lemma 2.5, Lemma 2.6 and (2.4). \square

From now on we assume that $i \geq 1$. Define a linear transformation $\tau_v : C^i(X) \rightarrow C^{i-1}(X)$ by

$$(\tau_v f)(s) = \begin{cases} f([v, s]) & \text{if } s \in S_{i-1}(\text{Lk}(v)); \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.8. *For $f, g \in C^i(X)$, we have $(\tau_v f, \tau_v g)_v = (\rho_v f, \rho_v g)$.*

Proof. We have

$$(\tau_v f, \tau_v g)_v = \sum_{\sigma \in \widehat{S}_{i-1}(\text{Lk}(v))} w_v(\sigma) \cdot \tau_v f(\sigma) \cdot \tau_v g(\sigma).$$

It is easy to see that there is a one-to-one correspondence between the n -simplices of X containing $[v, \sigma]$ and the $(n-1)$ -simplices of $\text{Lk}(v)$ containing σ , so $w_v(\sigma) = w([v, \sigma])$. Hence the above sum can be rewritten as

$$\sum_{s \in \widehat{S}_i(\text{St}(v))} w(s) \cdot (\rho_v f)(s) \cdot (\rho_v g)(s).$$

Since $\rho_v f$ is zero away from $\text{St}(v)$, the sum can be extended to the whole $\widehat{S}_i(X)$, so the lemma follows. \square

Lemma 2.9. *For $f \in C^i(X)$, we have $(\Delta \rho_v f, \rho_v f) = (\Delta_v \tau_v f, \tau_v f)_v$.*

Proof. By Lemma 2.5 and Lemma 2.8,

$$(\Delta \rho_v f, \rho_v f) = (\rho_v \Delta \rho_v f, \rho_v f) = (\tau_v \Delta \rho_v f, \tau_v f)_v.$$

Expanding $\tau_v \Delta \rho_v f(s)$ for $s \in S_{i-1}(\text{Lk}(v))$, one easily checks that $\tau_v \Delta \rho_v f = \Delta_v \tau_v f$, which implies the claim. \square

Lemma 2.10. *If c is an eigenvalue of Δ_v acting on $C^{i-1}(\text{Lk}(v))$ for some $v \in \text{Ver}(X)$, then c is also an eigenvalue of Δ acting on $C^i(X)$.*

Proof. Let $f \in C^{i-1}(\text{Lk}(v))$ be such that $\Delta_v f = c \cdot f$. Define $g \in C^i(X)$ as follows. If $s \in S_i(X)$ does not contain v then $g(s) = 0$. If $s = [v, \sigma]$ then $g(s) = f(\sigma)$. In particular, $\tau_v g = f$. We know that $\tau_v \Delta \rho_v g = \Delta_v \tau_v g$. Obviously $\rho_v g = g$, so $\tau_v \Delta g = \Delta_v f = c \cdot f = \tau_v(c \cdot g)$. This implies that $\Delta g = c \cdot g$. \square

Notation 2.11. Given a finite simplicial complex Y satisfying (1_X) , let $M^i(Y)$ and $m^i(Y)$ be the maximal and minimal non-zero eigenvalues of Δ acting on $C^i(Y)$, respectively. Denote

$$\lambda_{\max}^i(Y) := \max_{v \in \text{Ver}(Y)} M^i(\text{Lk}(v)) \quad \text{and} \quad \lambda_{\min}^i(Y) := \min_{v \in \text{Ver}(Y)} m^i(\text{Lk}(v)).$$

Corollary 2.12. $M^i(X) \geq \lambda_{\max}^{i-1}(X)$ and $m^i(X) \leq \lambda_{\min}^{i-1}(X)$.

Proposition 2.13. *For $f \in C^i(X)$, we have*

$$(\Delta \rho_v f, \rho_v f) \leq \lambda_{\max}^{i-1}(X) \cdot (\rho_v f, f).$$

Proof. By Lemma 2.9, $(\Delta \rho_v f, \rho_v f) = (\Delta_v \tau_v f, \tau_v f)_v$. Let $\{e_1, \dots, e_h\}$ be an orthogonal basis of $C^{i-1}(\text{Lk}(v))$ with respect to $(\cdot, \cdot)_v$ which consists of Δ_v -eigenvectors. Write $\tau_v f = \sum_j a_j e_j$. Then

$$(\Delta_v \tau_v f, \tau_v f)_v \leq M^{i-1}(\text{Lk}(v)) \sum_{j=1}^h a_j^2 (e_j, e_j)_v \leq \lambda_{\max}^{i-1}(X) \cdot (\tau_v f, \tau_v f)_v.$$

On the other hand, by Lemma 2.5 and Lemma 2.8, $(\tau_v f, \tau_v f)_v = (\rho_v f, \rho_v f) = (\rho_v f, f)$. \square

Denote by $\tilde{H}^i(\text{Lk}(v), \mathbb{R})$ the i th reduced simplicial cohomology group of $\text{Lk}(v)$.

Theorem 2.14 (Fundamental Inequality). *For $1 \leq i \leq n-1$, we have*

$$i \cdot M^i(X) \leq (i+1) \cdot \lambda_{\max}^{i-1}(X) - (n-i).$$

If $\tilde{H}^{i-1}(\text{Lk}(v), \mathbb{R}) = 0$ for every $v \in \text{Ver}(X)$, then

$$i \cdot m^i(X) \geq (i+1) \cdot \lambda_{\min}^{i-1}(X) - (n-i).$$

Proof. Let $f \in C^i(X)$ be such that $\Delta f = M^i(X) \cdot f$. Proposition 2.13 implies that the assumption of Corollary 2.7 is satisfied with $\Lambda = \lambda_{\max}^{i-1}(X)$. This proves the first part. The second part is Garland's original fundamental estimate [7, §5]. For a proof see Theorem 1.5 in [2]. \square

Notation 2.15. For $m \geq 1$, let I_m denote the $m \times m$ identity matrix and let J_m denote the $m \times m$ matrix whose entries are all equal to 1. The minimal polynomial of J_m is $x(x-m)$.

Example 2.16. Let X be an n -simplex. We claim that the eigenvalues of Δ acting on $C^i(X)$ are 0 and $(n+1)$ for any $0 \leq i \leq n-1$. It is easy to see that 0 is an eigenvalue, so we need to show that the only non-zero eigenvalue of Δ is $(n+1)$, or equivalently, $m^i(X) = M^i(X) = n+1$. First, suppose $i = 0$. Since for any simplex of X there is a unique n -simplex containing it, one easily checks that Δ acts on $C^0(X)$ as the matrix $(n+1)I_{n+1} - J_{n+1}$. The only eigenvalues of this matrix are 0 and $(n+1)$. Now let $i \geq 1$. The link of any vertex is an $(n-1)$ -simplex, so by induction $\lambda_{\min}^{i-1}(X) = \lambda_{\max}^{i-1}(X) = n$. Since the reduced cohomology groups of a simplex vanish, the Fundamental Inequality implies

$$i \cdot M^i(X) \leq (i+1)n - (n-i) = i(n+1)$$

and

$$i \cdot m^i(X) \geq (i+1)n - (n-i) = i(n+1).$$

Hence $(n+1) \leq m^i(X) \leq M^i(X) \leq (n+1)$, which implies the claim.

2.4. Buildings. Let G be a group equipped with a Tits system (G, B, N, S) of rank $\ell+1$. To every Tits system, there is an associated simplicial complex \mathfrak{B} of dimension ℓ , called the *building* of (G, B, N, S) . For the definitions and basic properties of buildings we refer to Chapters IV and V in [3]. The simplices of \mathfrak{B} are in one-to-one correspondence with proper parabolic subgroups of G . Assume from now on that G is finite. Then \mathfrak{B} is a finite simplicial complex satisfying (1_X) . Given a simplex s of \mathfrak{B} , it is known that $\text{Lk}(s)$ is again a building corresponding to a Tits system of rank $\ell - \dim(s)$.

We would like to estimate $M^i(\mathfrak{B})$ and $m^i(\mathfrak{B})$ for $0 \leq i \leq \ell - 1$. This will be done inductively, using induction on i and ℓ . The base of induction is the following lemma:

Lemma 2.17. *If $\ell = 1$ then $M^0(\mathfrak{B}) = 2$, and $m^0(\mathfrak{B}) \leq 1$.*

Proof. When $\ell = 1$, the eigenvalues of Δ acting on $C^0(\mathfrak{B})$ were calculated by Feit and Higman in [6]. The claim follows from these calculations. See also Proposition 7.10 in [7] when \mathfrak{B} is of Lie type. \square

Let K be the fundamental chamber of \mathfrak{B} , i.e., the ℓ -simplex of \mathfrak{B} corresponding to the Borel subgroup B of the given Tits system. Every simplex s of \mathfrak{B} can be transformed to a unique face s' of K under the action of G . Label the vertices of K by the elements of $I_\ell := \{0, 1, \dots, \ell\}$, and define $\text{Type}(s)$ to be the subset of I_ℓ corresponding to the vertices of s' . G naturally acts on \mathfrak{B} and this action is type-preserving and strongly transitive; see [3, §V.3]. From this perspective one can think of K as the quotient \mathfrak{B}/G .

Lemma 2.18. *$\ell + 1$ is an eigenvalue of Δ acting on $C^i(\mathfrak{B})$. In particular, $M^i(\mathfrak{B}) \geq \ell + 1$.*

Proof. Given a function $f \in C^i(K)$, we can lift it (uniquely) to a G -invariant function $\tilde{f} \in C^i(\mathfrak{B})$ defined by $\tilde{f}(\tilde{s}) := f(s)$, where \tilde{s} is any preimage of s in \mathfrak{B} . As is explained in [2, §4.2], we have $\Delta f = \Delta \tilde{f}$. Hence the claim follows from Example 2.16. \square

Remark 2.19. Note that Lemma 2.10, coupled with Lemma 2.18, implies that the integers $\ell + 1, \ell, \dots, \ell - i + 1$ are always present among the eigenvalues of Δ acting on $C^i(\mathfrak{B})$. On the contrary, $\ell - i$ is not necessarily an eigenvalue.

Let $f \in C^0(\mathfrak{B})$, and let R be a fixed constant. For each $\alpha \in I_\ell$, define a function f_α on the vertices of \mathfrak{B} by $f_\alpha(v) = f(v)$ if $\text{Type}(v) \neq \alpha$ and $f_\alpha(v) = R \cdot f(v)$ if $\text{Type}(v) = \alpha$. Also, for a fixed $\alpha \in I_\ell$ define a linear transformation ρ_α on $C^0(\mathfrak{B})$ by

$$\rho_\alpha = \sum_{\text{Type}(v)=\alpha} \rho_v.$$

For $f \in C^0(\mathfrak{B})$ and any α , we have

$$(2.6) \quad (\rho_\alpha df_\alpha, df_\alpha) = (df_\alpha, df_\alpha) - ((1 - \rho_\alpha)df, df),$$

and

$$(2.7) \quad (\Delta \rho_\alpha df_\alpha, \rho_\alpha df_\alpha) = ((1 - \rho_\alpha)df, df)$$

The equations (2.6) and (2.7) are the equations (3) and (6) in [2, §4.5], respectively.

Lemma 2.20. *Let $f \in C^0(\mathfrak{B})$ and suppose $\Delta f = c \cdot f$. Then*

$$\sum_{\alpha \in I_\ell} (\Delta f_\alpha, f_\alpha) = [(\ell - c)(R - 1)^2 + c(R^2 + \ell)] \cdot (f, f).$$

Proof. Fix some type α and let $g \in C^0(\mathfrak{B})$ be a function such that $g(v) = 0$ if $\text{Type}(v) \neq \alpha$. Then $(\Delta g, g) = \ell \cdot (g, g)$. Indeed,

$$\begin{aligned} (\Delta g, g) &= (dg, dg) = \sum_{[x, v] \in \widehat{S}_1(\mathfrak{B})} w([x, v])(g(v) - g(x))^2 \\ &= \sum_{\text{Type}(v)=\alpha} g(v)^2 \sum_{x \in \text{Ver}(\text{Lk}(v))} w([x, v]) = \ell \sum_{\text{Type}(v)=\alpha} w(v) \cdot g(v)^2 = \ell \cdot (g, g). \end{aligned}$$

(The middle equality on the previous line follows from Lemma 2.1.) If we apply this to $g = f_\alpha - f$, then we get

$$(2.8) \quad (\Delta f_\alpha, f_\alpha) = \ell \cdot (f_\alpha, f_\alpha) - 2(\ell - c)(f_\alpha, f) + (\ell - c)(f, f).$$

We clearly have

$$\sum_{\alpha \in I_\ell} f_\alpha = (\ell + R) \cdot f \quad \text{and} \quad \sum_{\alpha \in I_\ell} (f_\alpha, f_\alpha) = (\ell + R^2) \cdot (f, f).$$

Summing (2.8) over all types and using the previous two equalities, we get the claim. \square

Theorem 2.21. $M^i(\mathfrak{B}) = \ell + 1$.

Proof. By Lemma 2.18, it is enough to show that $M^i(\mathfrak{B}) \leq \ell + 1$. We start with $M^0(\mathfrak{B})$. Let $f \in C^0(\mathfrak{B})$. Since the vertices of any simplex in \mathfrak{B} have distinct types, one easily checks that

$$\sum_{\text{Type}(v)=\alpha} (\Delta \rho_v df_\alpha, \rho_v df_\alpha) = (\Delta \rho_\alpha df_\alpha, \rho_\alpha df_\alpha),$$

so by Proposition 2.13

$$(2.9) \quad (\Delta \rho_\alpha df_\alpha, \rho_\alpha df_\alpha) \leq \lambda_{\max}^0(\mathfrak{B}) \cdot (\rho_\alpha df_\alpha, \rho_\alpha df_\alpha).$$

Since for any $v \in \text{Ver}(\mathfrak{B})$, $\text{Lk}(v)$ is a building of dimension $\ell - 1$, the induction on ℓ gives $\lambda_{\max}^0(\mathfrak{B}) = \ell$. Combining this with (2.9), (2.6) and (2.7), we get

$$(2.10) \quad (1 + \ell) \cdot ((1 - \rho_\alpha)df, df) \leq \ell \cdot (df_\alpha, df_\alpha).$$

Now assume $\Delta f = c \cdot f$. Note that

$$(2.11) \quad \begin{aligned} \sum_{\alpha \in I_\ell} (1 - \rho_\alpha)df &= (\ell + 1)df - \sum_{v \in \text{Ver}(\mathfrak{B})} \rho_v df \\ &= (\ell + 1)df - 2df = (\ell - 1)df, \end{aligned}$$

so summing the inequalities (2.10) over all types and using Lemma 2.20, we get

$$(2.12) \quad (\ell + 1)(\ell - 1)c \cdot (f, f) \leq \ell \cdot [(\ell - c)(R - 1)^2 + c(R^2 + \ell)] \cdot (f, f).$$

If we put $R = (\ell - c)/\ell$, then (2.12) forces $c \leq \ell + 1$. In particular, $M^0(\mathfrak{B}) \leq \ell + 1$.

Now let $i \geq 1$. The induction on i and ℓ implies that $\lambda_{\max}^{i-1}(\mathfrak{B}) = \ell$. From the Fundamental Inequality 2.14 we get

$$i \cdot M^i(\mathfrak{B}) \leq (i + 1) \cdot \ell - (\ell - i),$$

which implies $M^i(\mathfrak{B}) \leq \ell + 1$. \square

Theorem 2.22. $m^i(\mathfrak{B}) \leq \ell - i$.

Proof. We start with $i = 0$. Denote $c := m^0(\mathfrak{B})$ and let f be a Δ -eigenfunction with eigenvalue c . First we claim that $c \neq \ell + 1$. Indeed, Δ is a semi-simple operator and if $c = \ell + 1$ then by Theorem 2.21 it has only two distinct eigenvalues, namely 0 and $\ell + 1$. This implies that $\Delta^2 = (\ell + 1)\Delta$. But it is easy to check that this equality is false. Next, $c \neq \ell + 1$ implies $(\Delta f_\alpha, f_\alpha) \geq c \cdot (f_\alpha, f_\alpha)$; see equation (1) in [2, §4.6]. Summing over all types,

$$\sum_{\alpha \in I_\ell} (\Delta f_\alpha, f_\alpha) \geq c(\ell + R^2) \cdot (f, f).$$

Comparing this inequality with the expression in Lemma 2.20, we conclude that

$$(\ell - c)(R - 1)^2 \geq 0.$$

Since R is arbitrary, we must have $c \leq \ell$.

Now assume $i \geq 1$. By Corollary 2.12 and induction on i and ℓ , we have $m^i(\mathfrak{B}) \leq \lambda_{\min}^{i-1}(\mathfrak{B}) \leq (\ell - 1) - (i - 1) = \ell - i$. \square

Theorem 2.23 (Garland). *Assume that G is the group of \mathbb{F}_q -valued points of a simple, simply connected Chevalley group. For any $\varepsilon > 0$ there is a constant $q(\varepsilon, \ell)$ depending only on ε and ℓ , such that if $q > q(\varepsilon, \ell)$ then $m^i(\mathfrak{B}) \geq \ell - i - \varepsilon$.*

Proof. For the proof see Sections 6, 7, 8 in [7], or Proposition 5.4 in [2]. \square

3. EXAMPLES

In this section we compute explicitly in some cases the eigenvalues of Δ acting on $C^i(\mathfrak{B})$. We concentrate on $G = \mathrm{SL}_{\ell+2}(\mathbb{F}_q)$ for small ℓ , with $B \subset G$ being the upper triangular group and N being the monomial group, cf. [3, §V.5]. Denote the corresponding building by $\mathfrak{B}_{\ell,q}$. The dimension of $\mathfrak{B}_{\ell,q}$ is ℓ . Denote by $\mathfrak{m}_\ell^i(q; x)$ the minimal polynomial of Δ acting on $C^i(\mathfrak{B}_{\ell,q})$, $0 \leq i \leq \ell - 1$.

First, we recall an elementary description of $\mathfrak{B}_{\ell,q}$ which is convenient for actual calculations. Let V be a linear space over \mathbb{F}_q of dimension $\ell + 2$. A *flag* in V is a nested sequence $\mathcal{F} : F_0 \subset F_1 \subset \cdots \subset F_i$ of distinct linear subspaces F_0, \dots, F_i of V such that $F_0 \neq 0$ and $F_i \neq V$. $\mathfrak{B}_{\ell,q}$ is isomorphic to the simplicial complex whose vertices correspond to the non-zero linear subspaces of V distinct from V ; the vertices v_0, \dots, v_i form an i -simplex if the corresponding subspaces form a flag.

Now assume $\ell = 1$. In this case $\mathfrak{B}_{\ell,q}$ is isomorphic to the 1-dimensional complex whose vertices correspond to 1 and 2-dimensional subspaces of a 3-dimensional vector space V over \mathbb{F}_q , two vertices being adjacent if one of the corresponding subspaces is contained in the other. With a slight abuse of terminology, we will call 1 and 2 dimensional subspaces lines and planes, respectively. The number of lines and planes in V is $m = q^2 + q + 1$ each. Let $A = (a_{ij})$ be the $m \times m$ matrix whose rows are enumerated by the lines in V and columns by the planes, and $a_{ij} = -1$ if the i th line lies in the j th plane, and is 0 otherwise. We can choose a basis of $C^0(\mathfrak{B}_{\ell,q})$ so that $(q + 1)\Delta$ acts as the matrix

$$(q + 1)I_{2m} + \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}.$$

Let $M = \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$. Since any two distinct lines lie in a unique plane and any line lies in $(q + 1)$ planes, $AA^t = qI_m + J_m$. By a similar argument, $A^tA = qI_m + J_m$. Hence

$$M^2 = qI_{2m} + \begin{pmatrix} J_m & 0 \\ 0 & J_m \end{pmatrix}.$$

This implies that $(M^2 - qI_{2m})(M^2 - (q + 1)^2I_{2m}) = 0$. Since $(q + 1)\Delta - (q + 1)I_{2m} = M$, we conclude that $(q + 1)\Delta$ satisfies the polynomial equation

$$x(x - (2q + 2))(x^2 - (2q + 2)x + (q^2 + q + 1)) = 0.$$

It is not hard to see that this is in fact the minimal polynomial of $(q + 1)\Delta$. Hence

$$\mathfrak{m}_1^0(q; x) = x(x - 2) \left(x^2 - 2x + \frac{q^2 + q + 1}{q^2 + 2q + 1} \right).$$

The minimal non-zero root is $1 - \sqrt{q}/(q + 1)$. The smallest possible value of this expression is approximately 0.53, which occurs at $q = 2$, the value tends to 1 as $q \rightarrow \infty$.

The very next case $\ell = 2$ is already considerably harder to compute by hand. With the help of a computer, we deduced that

$$\begin{aligned} \mathfrak{m}_2^0(q; x) = & x(x-2)(x-3) \left(x - \frac{2q^2 + 3q + 2}{q^2 + q + 1} \right) \\ & \times \left(x^2 - \frac{4q^2 + 3q + 4}{q^2 + q + 1}x + \frac{4q^2 + 4}{q^2 + q + 1} \right). \end{aligned}$$

The minimal non-zero root is

$$\frac{1}{2(q^2 + q + 1)} \left(4q^2 + 3q + 4 - \sqrt{8q^3 + 9q^2 + 8q} \right),$$

which is at least 1.08 and tends to 2 from below as $q \rightarrow \infty$. The whole polynomial tends coefficientwise to the polynomial $x(x-3)(x-2)^4$ as $q \rightarrow \infty$. Next

$$\begin{aligned} \mathfrak{m}_2^1(q; x) = & x(x-1)(x-2)(x-3) \\ & \times \left(x^2 - 2x + \frac{q^2 + 1}{q^2 + 2q + 1} \right) \left(x^2 - 3x + \frac{2q^2 + 2q + 2}{q^2 + 2q + 1} \right) \\ & \times \left(x^2 - 4x + \frac{4q^2 + 6q + 4}{q^2 + 2q + 1} \right). \end{aligned}$$

The minimal non-zero root is $1 - \sqrt{2q}/(q+1)$. This is always in the interval $[1/3, 1)$. Moreover, this eigenvalue is strictly larger than $1/3$ for $q > 2$ and tends to 1 as $q \rightarrow \infty$; the whole polynomial tends to $x(x-3)(x-1)^4(x-2)^4$.

The formulae for $\mathfrak{m}_2^0(q; x)$ and $\mathfrak{m}_2^1(q; x)$ are partly conjectural, although almost certainly correct. We computed these polynomials for $q = 2, 3, 4, 5, 7$ using computer calculations with concrete finite fields, and then came up with a formula which recovers all the previous polynomials when we specialize q .

The complexity of calculations grows exponentially with i , ℓ and q , so for $\ell = 3$ my computer was able to handle only $i = 0$ for $q = 2$ and 3:

$$\begin{aligned} \mathfrak{m}_3^0(2; x) = & x(x-4) \left(x - \frac{23}{7} \right) \left(x - \frac{19}{7} \right) \\ & \times \left(x^4 - 12x^3 + \frac{581528}{11025}x^2 - \frac{220232}{2205}x + \frac{6734719}{99225} \right), \end{aligned}$$

$$\begin{aligned} \mathfrak{m}_3^0(3; x) = & x(x-4) \left(x - \frac{42}{13} \right) \left(x - \frac{36}{13} \right) \\ & \times \left(x^4 - 12x^3 + \frac{14350977}{270400}x^2 - \frac{2760633}{27040}x + \frac{309843369}{4326400} \right). \end{aligned}$$

The minimal non-zero roots of these polynomials are approximately 1.68 and 1.89, respectively. To have a reasonable guess for the coefficients of $\mathfrak{m}_3^0(q; x)$, one needs to compute these polynomials for at least the next few values of q . Nevertheless, note that the coefficients of above polynomials are close to the coefficients of $x(x-4)(x-3)^6$.

The final example we have is

$$\begin{aligned} \mathfrak{m}_4^0(2; x) = & x(x-4)(x-5) \left(x - \frac{144}{35} \right) \left(x^2 - \frac{1322}{155}x + \frac{2798}{155} \right) \\ & \times \left(x^2 - \frac{276}{35}x + \frac{536}{35} \right) \left(x^3 - \frac{1778}{155}x^2 + \frac{1306}{31}x - \frac{7512}{155} \right). \end{aligned}$$

The minimal non-zero root is approximately 2.32, and the coefficients of $\mathfrak{m}_4^0(2; x)$ are close to the coefficients of $x(x-5)(x-4)^9$.

The previous calculations, combined with Theorems 2.21-2.23 and Remark 2.19, suggest the following possibility:

Conjecture 3.1. In the situation of Theorem 2.23, for any $\varepsilon > 0$ there is a constant $q(\varepsilon, \ell)$ depending only on ℓ and ε such that if $q > q(\varepsilon, \ell)$ then any non-zero eigenvalue of Δ acting on $C^i(\mathfrak{B})$ is at a distance less than ε from one of the integers

$$\ell - i, \ell - i + 1, \dots, \ell + 1.$$

3.1. Garland's vanishing theorem. Let \mathcal{K} be a field complete with respect to a non-trivial discrete valuation and which is locally compact. Let \mathbb{F}_q be the residue field of \mathcal{K} . Let \mathcal{G} be an almost simple linear algebraic group over \mathcal{K} . Suppose \mathcal{G} has \mathcal{K} -rank $\ell + 1$. Let \mathfrak{T} be the Bruhat-Tits building associated with $\mathcal{G}(\mathcal{K})$. The link of a simplex s in \mathfrak{T} is a finite building of dimension $\ell - \dim(s)$. Using a discrete analogue of Hodge decomposition and the Fundamental Inequality one proves the following theorem (see [2, Thm. 3.3]):

Theorem 3.2. If $\lambda_{\min}^{i-1}(\mathfrak{T}) > \frac{\ell+1-i}{i+1}$, then $H^i(\Gamma, \mathbb{R}) = 0$ for any discrete cocompact subgroup Γ of $\mathcal{G}(\mathcal{K})$.

Combining this with Theorem 2.23, one concludes that there is a constant $q(\ell)$ depending only on ℓ such that if $q > q(\ell)$ then $H^i(\Gamma, \mathbb{R}) = 0$ for $1 \leq i \leq \ell$. This is the main result of [7]. It is natural to ask whether the restriction on q being sufficiently large is redundant. This is indeed the case, as was shown by Casselman [5], who proved the vanishing of the middle cohomology groups by an entirely different argument.

Now let $\mathcal{G} = \mathrm{SL}_{\ell+2}$. Then $\lambda_{\min}^{i-1}(\mathfrak{T}) = m^{i-1}(\mathfrak{B}_{\ell,q})$. In all examples discussed above $m^0(\mathfrak{B}_{\ell,q}) > \ell/2$, so in these cases Garland's method proves the vanishing of $H^1(\Gamma, \mathbb{R})$ without any assumptions on q . On the other hand, $m^1(\mathfrak{B}_{2,2}) = 1/3$. But to apply Theorem 3.2 to show that $H^2(\Gamma, \mathbb{R}) = 0$ we need $\lambda_{\min}^1(\mathfrak{T}) > 1/3$. Hence when $\ell = 2$ we need to assume $q > 2$ to conclude $H^2(\Gamma, \mathbb{R}) = 0$ from Garland's method.

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